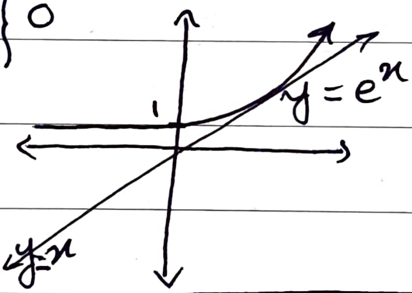
Lec 02 No. Date. Jun 23, 2026

Review: $\lim_{x \to a} f(x) = L$ For every $\varepsilon > 0$, there's a $\delta > 0$ such that if $|x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

★ Today's pun: Sunday was the summer soltice, so now the days are shortening. We hope for butter days ahead...

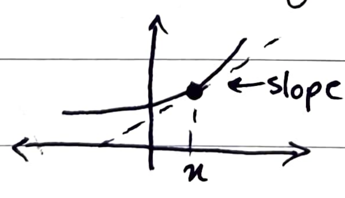
$$\lim_{x \to -\infty} \frac{e^x \to 0^+}{x \to -\infty} \Big\} 0^-$$



→ The numerator is + & the denominator is − & getting smaller.

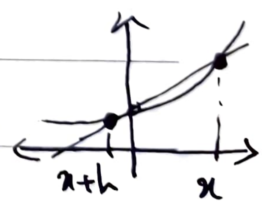
Derivatives: $y = f(x)$ $\frac{dy}{dx} = f'(x) = \dot{y}$

the slope of



How do we define derivatives?

$$\text{The slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$ so for



$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

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officially: $\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

so $\frac{d}{dx} f(x) = y'$

examples:

$f(x) = x$

$f'(3) = \lim_{h \to 0} \frac{(3+h)-3}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$

$f(x) = x^2$

$\frac{dy}{dx} = \frac{d}{dx}(x^2) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \Rightarrow$

$\lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h} \Rightarrow$

$\lim_{h \to 0} 2x + h = \lim_{h \to 0} 2x + 0 = 2x$

$y = e^x$

$\frac{dy}{dx} = \frac{d}{dx} e^x = \lim_{h \to 0} \left(\frac{e^{x+h} - e^x}{h} \right) \Rightarrow$

$\lim_{h \to 0} \left(\frac{e^x e^h - e^x}{h} \right) \Rightarrow \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} \Rightarrow$

$e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x(1) = e^x$

⋆ e^x is a constant because $\lim_{h \to 0}$ looks at h & not necessarily x, so we can multiple it.

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$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \Rightarrow$$

$$\lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \Rightarrow$$

$$\lim_{h\to 0}\frac{\sin(x)[\cos(h)-1]+\cos(x)\sin(h)}{h} \Rightarrow$$

$$\left(\lim_{h\to 0}\frac{\sin(x)[\cos(h)-1]}{h}\right)+\left(\lim_{h\to 0}\frac{\cos(x)\sin(h)}{h}\right) \Rightarrow$$

$$\sin(x)\cdot\lim_{h\to 0}\frac{[\cos(h)-1]}{h}+\cos(x)\cdot\lim_{h\to 0}\frac{\sin(h)}{h} \Rightarrow$$

$$\sin(x)(0)+\cos(x)(1) \Rightarrow \cos(x)$$

★ Similarly: $\frac{d}{dx}\cos(x) = -\sin(x)$

Practical Rules for Derivatives:

1° (Constant Rule) $\frac{d}{dx}(c\,f(x)) = c\left(\frac{d}{dx}f(x)\right) =$

$cf'(x)$

$(cf)'(x)$

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2° (Sum Rule)

$$\frac{d}{dx}(f(x)+g(x)) = \left(\frac{d}{dx}f(x)\right)+\left(\frac{d}{dx}g(x)\right)$$

ie. $(f+g)'(x) = f'(x)+g'(x)$

3° (Power Rule)

$$\frac{d}{dx}x^n = nx^{n-1}$$

ie. $\frac{d}{dx}x^2 = 2x^{2-1} = 2x$

4° (Product Rule)

$$\frac{d}{dx}(f(x)\cdot g(x)) = \left[\frac{d}{dx}f(x)\right]\cdot g(x) + f(x)\left[\frac{d}{dx}g(x)\right]$$

or $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

5° (chain Rule)

$$\frac{d}{dx}(f(g(x))) = f'(g(x))\cdot g'(x)$$

or $(f\circ g)'(x)$

6° (Quotient Rule)

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\left[\frac{d}{dx}f(x)\right]\cdot g(x) - f(x)\cdot\left[\frac{d}{dx}g(x)\right]}{[g(x)]^2}$$

$$\text{or } \left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

★ Proof of the product rule (4°):

$$\frac{d}{dx}(f(x)g(x)) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h} \Rightarrow$$

$$\lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x+h) - f(x)g(x)}{h} \Rightarrow$$

$$\lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h\to 0}\frac{f(x)g(x+h) - f(x)g(x)}{h} \Rightarrow$$

$$\lim_{h\to 0} g(x+h)\cdot\frac{f(x+h)+f(x)}{h} + \lim_{h\to 0} f(x)\cdot\frac{g(x+h)-g(x)}{h} \Rightarrow$$

$$\left[\lim_{h\to 0} g(x+h)\right]\left[\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right] + f(x)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \Rightarrow$$

$$g(x)\cdot f'(x) + f(x)\cdot g'(x) \Rightarrow$$

$$\boxed{f'(x)\cdot g(x) + f(x)\cdot g'(x)}$$

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best number!

exp. $\frac{d}{dx}(x^2-4)^{41} \Rightarrow 41(x^2-4)^{40}\cdot(2x-0) \Rightarrow$

$\boxed{82x(x^2-4)^{40}}$

we used:
1. chain rule
2. product rule
3. sum & constant rule

Basic ~~data~~ derivatives to memorize:

0° $\frac{d}{dx}x^n = nx^{n-1}$ (power rule)

1° $\frac{d}{dx}\sin(x) = \cos(x)$ & $\frac{d}{dx}\cos(x) = -\sin(x)$

$\frac{d}{dx}\tan(x) = \sec^2(x)$ & $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$

$\rightarrow \frac{d}{dx}\tan(x) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{[\frac{d}{dx}\sin(x)]\cos(x) - \sin(x)[\frac{d}{dx}\cos(x)]}{\cos^2(x)}$

$\Rightarrow \frac{\cos(x)\cos(x)+\sin(x)\sin(x)}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$

$\Rightarrow \left(\frac{1}{\cos(x)}\right)^2 = \sec^2(x)$

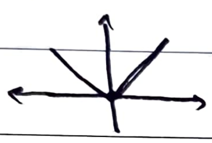
2° $\frac{d}{dx}\ln(x) = \frac{1}{x}$

$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$ } because: $\log_a(x) = \frac{\ln(x)}{\ln(a)}$

3° $\frac{d}{dx} e^x = e^x$

$\frac{d}{dx} a^x = \ln(a) a^x$ because: $a^x = (e^{\ln(a)})^x = e^{\ln(a) \cdot x}$

4° $y = |x|$ $\Rightarrow \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases}$



so $\frac{dy}{dx} = \begin{cases} 1 & x > 0 \\ \text{undefined} & x = 0 \\ -1 & x < 0 \end{cases}$

5° "hyperbolic functions"

$\sinh(x) = \frac{e^x - e^{-x}}{2}$ ("Sinch")

$\cosh(x) = \frac{e^x + e^{-x}}{2}$ ("Kosh")

$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$

$\mathrm{sech}(x) = \frac{1}{\cosh(x)}$

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Practice :

$$y = \frac{e^{x+\sin(x)}}{x+3}$$

$$\frac{dy}{dx} = \frac{\left[\frac{d}{dx} e^{x+\sin(x)}\right](x+3) - e^{x+\sin(x)}\left[\frac{d}{dx}(x+3)\right]}{(x+3)^2} \Rightarrow$$

$$\frac{dy}{dx} = \frac{e^{x+\sin(x)} \cdot \left[\frac{d}{dx}(x+\sin(x))\right] \cdot (x+3) - e^{x+\sin(x)} \cdot 1}{(x+3)^2} \Rightarrow$$

$$\frac{dy}{dx} = \frac{e^{x+\sin(x)} \cdot \left[1+\cos(x)\right](x+3) - e^{x+\sin(x)}}{(x+3)^2} \Rightarrow$$

$$\frac{e^{x+\sin(x)}\left[(1+\cos(x))(x+3)-1\right]}{(x+3)^2} \Rightarrow$$

$$\frac{e^{x+\sin(x)}\left[x + x\cos(x) + \overset{2}{3\cancel{-1}}\right]}{(x+3)^2} \Rightarrow$$

$$\boxed{\frac{e^{x+\sin(x)}\left[x + x\cos(x) + 2\right]}{(x+3)^2}}$$